Dynamic Interpolation and Higher-Order Turbulence Scales in 3D Navier–Stokes Flows

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Outline

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The Hyperdissipative Navier–Stokes System

We study the three-dimensional hyperdissipative Navier–Stokes system on $\mathbb{R}^3 \times (0, T)$:

$$\begin{cases} \partial_t u + (-\Delta)^\beta u + (u \cdot \nabla) u + \nabla p = 0, \\ \nabla \cdot u = 0, \\ u(\cdot, 0) = u_0(\cdot). \end{cases} \tag{1}$$

- $(-\Delta)^\beta$ acts in Fourier space by $|\xi|^{2\beta}$,
- $\beta > 1$ controls the strength of dissipation.

The nonlinearity can be written in divergence form:

$$(u \cdot \nabla) u = \nabla \cdot (u \otimes u).$$

Scaling and Criticality

Under the scaling

$$u_\lambda(x,t) = \lambda^{2\beta-1} u(\lambda x, \lambda^{2\beta} t), \qquad \|u_\lambda(t)\|_2 = \lambda^{2\beta-5/2} \left\| u(\lambda^{2\beta} t) \right\|_2$$

the system (1) is invariant.

Criticality classification in 3D

- **Subcritical:** $\beta > \frac{5}{4}$
- **Critical:** $\beta = \frac{5}{4}$
- **Supercritical:** $1 \leq \beta < \frac{5}{4}$

Key point: $\beta = \frac{5}{4}$ is the Lions threshold, where the energy norm becomes scale invariant.

The Scaling Gap in Classical 3D Navier–Stokes

What the energy estimate gives

Leray–Hopf solutions satisfy

$$u \in L^p(0, T; L^q(\mathbb{R}^3)), \qquad \frac{2}{p} + \frac{3}{q} = \frac{3}{2}, \qquad 2 \le q \le \infty.$$

What guarantees regularity

The Ladyzhenskaya–Prodi–Serrin criterion applies when

$$u \in L^p(0, T; L^q(\mathbb{R}^3)), \qquad \frac{2}{p} + \frac{3}{q} = 1, \qquad 3 < q \le \infty.$$

At $q = 6$, the gap is explicit. Energy gives $u \in L^2_t L^6_x$, while LPS requires $u \in L^4_t L^6_x$.

Problem and Main Idea

Unconditional global regularity is open for

$$1 < \beta < \frac{5}{4}.$$

Recent progress suggests that regularity below the Lions threshold may depend on:

- **analyticity scales**,
- **geometric sparseness of intense regions**,
- **harmonic measure arguments**.

Main contributions

We quantify the **analyticity–sparseness gap** by introducing a **time-weighted bridges** across derivative levels, which allows **flexibility** in the argument, thus ruling out a class of strongly turbulent blow-up scenarios

1D δ-Sparseness at derivative level k

Definition

Let $S \subset \mathbb{R}^3$, $\delta \in (0,1)$, $x_0 \in \mathbb{R}^3$, and $r > 0$. We say that S is *1-dimensional δ-sparse around x_0 at scale r* if there exists a unit vector $\nu \in S^2$ such that

$$\frac{|S \cap (x_0 - r\nu,\ x_0 + r\nu)|}{2r} \leq \delta.$$

In applications, S is a super-level set such as

$$S = \left\{ x \in \mathbb{R}^3 : \ |D^k u(x,t)| > \lambda \left\| D^k u(t) \right\|_\infty \right\}.$$

Interpretation

Intense fluid activity can be large in amplitude, while remaining thin along some direction.

Escape Times and Harmonic Measure

Definition (Escape time)

A time $t \in (0, T^*)$ is an escape time for $D^k u$ if

$$\left\| D^k u(\tau) \right\| > \left\| D^k u(t) \right\| \qquad \text{for every } \; t < \tau < T^*).$$

Proposition (Ransford)

Let Ω be an open, connected set in $\mathbb{C}$ such that its boundary has nonzero Hausdorff dimension, and let K be a Borel subset of the boundary. Suppose that u is a subharmonic function on Ω satisfying

$$u(z) \leq M, \quad \text{for } z \in \Omega$$

and

$$\limsup_{z \to \xi} u(z) \leq m, \quad \text{for } \xi \in K.$$

Then

$$u(z) \leq mh(z, \Omega, K) + M(1 - h(z, \Omega, K)), \quad \text{for } z \in \Omega.$$

The Existing Analyticity–Sparseness Mechanism

A central idea in the Grujić–Xu framework is:

Regularity criterion

At an escape time, if regions where a high-order derivative is large are sparse on a scale within the radius of analyticity, then the solution stays regular through the candidate blow-up time.

This naturally leads to the key comparison:

$$\rho_k \geq r_k,$$

where

- ρ_k is the analyticity radius,
- r_k is the sparse scale.

Comparing the Analyticity Radius with the Sparse Scale

$$r_k \gtrsim \|D^k u\|_\infty^{-\frac{1}{k+\frac{3}{2}}} \qquad \text{(sparse scale)}$$

Classical GN estimate: $\rho_k \gtrsim \|D^k u\|_\infty^{-\frac{1}{2\beta-1}\frac{3}{2}\frac{1}{k+\frac{3}{2}}}$

Time-independent bridge: $\rho_k \gtrsim \|D^k u\|_\infty^{-\frac{1}{2\beta-1}\frac{1}{k+1}}$

Time-weighted bridge: $\rho_k \gtrsim \|D^k u\|_\infty^{-a_k}, \qquad a_k = \dfrac{1}{2\beta-1-(\mu-1)\frac{2k}{k+1}} \cdot \dfrac{1}{k+1}$

Main point

The classical comparison reaches the sparse scale only above the critical threshold ($\beta > \frac{5}{4}$), whereas the weighted bridge (via **dynamic interpolation**) improves the radius enough to be useful already in the near-critical regime $1 < \beta < \frac{5}{4}$.

The Hyperdissipative Bridge: Time-Weighted Bridge Across Derivative Levels

A time-weighted bridge coefficient linking derivative levels:

$$\mathcal{M}_{j,k} = (T^* - t)^{-\frac{1}{2\beta}\left(\frac{\mu+j}{j+1} - \frac{\mu+k}{k+1}\right)} \frac{(j!)^{\frac{1}{j+1}}}{(k!)^{\frac{1}{k+1}}}.$$

This yields the schematic derivative comparison

$$\left\|D^j u(t)\right\|_\infty^{\frac{1}{j+1}} \lesssim \mathcal{M}_{j,k} \left\|D^k u(t)\right\|_\infty^{\frac{1}{k+1}}.$$

Since

$$\frac{\mu+j}{j+1} - \frac{\mu+k}{k+1} = (\mu-1)\left(\frac{1}{j+1} - \frac{1}{k+1}\right) \geq 0,$$

the time factor strengthens the ascending derivative structure as $t \uparrow T^*$.

Quantifying the Analyticity–Sparseness Gap

The analyticity radius has the schematic form

$$\rho_k \sim \left\| D^k u \right\|^{-a_k}, \qquad a_k = \frac{1}{2\beta - 1 - (\mu - 1)\frac{2k}{k+1}} \cdot \frac{1}{k+1}.$$

The sparse scale behaves like

$$r_k \sim \left\| D^k u \right\|^{-\frac{1}{k+\frac{3}{2}}}.$$

Define the exponent gap

$$\Delta_k := \frac{1}{k+\frac{3}{2}} - a_k.$$

Key observation

If $\Delta_k > 0$, then the analyticity radius dominates the sparse scale for sufficiently large k.

Positive Gap and Dominance of Scales

Clearly, for $1 < \beta < 5/4$ and $\mu < \beta$, we have $\Delta_k > 0$. Therefore

$$\rho_k(s) \geq c \left(\log \|u(s)\|_\infty\right)^{-a_k} \left\|D^k u(s)\right\|_\infty^{\Delta_k} r_k(s).$$

For sufficiently large k, this yields

$$\rho_k(s) \geq r_k(s).$$

Central consequence

The complex analytic tube contains the disks associated with sparse super-level sets, so the geometric information can be injected into the analytic argument.

APPLICATION

A Focused Taylor Blow-Up Scenario

Theorem (Phiri, 2026)

Let $u_0 \in L^2 \cap L^\infty$. Near a candidate isolated singularity x^* at T^*, assume

$$c^i_{\alpha,k}(t) = \delta^i_{\alpha,k}(t)\,\rho(t)^{-k},\ \rho(t) \to 0,\ \left\|D^{(k)}u(t)\right\|_\infty = |D^{(k)}u(x^*,t)|.$$

For $k \geq 2$, the modulation $\delta^i_{\alpha,k}(t)$ lie between $L_k(t)^{-1}$ and $L_k(t)$, where

$$L_k(t) := C^k(T^* - t)^{-\frac{\gamma k}{\log k}}.$$

Each relevant coefficient increase in time.

Conclusion

The solution extends spatially analytically beyond T^*

Remark: No assumptions or restrictions placed on ρ.

Sketch of Proof I

- Derive the time-weighted bridge

  Let $c^i_{\alpha,k}(t) = \frac{1}{\alpha!}\frac{\partial^k}{\partial^\alpha x} u^i(x^*, t)$ for $0 \le j \le k$. Then

$$\frac{\|D^j u(t)\|^{\frac{1}{j+1}}}{\|D^k u(t)\|^{\frac{1}{k+1}}} = \frac{|D^j u(t)|^{\frac{1}{j+1}}}{|D^k u(t)|^{\frac{1}{k+1}}} \le \frac{(j!)^{\frac{1}{j+1}}}{(k!)^{\frac{1}{k+1}}} \frac{(c^i_{\alpha,j})^{\frac{1}{j+1}}}{(c^i_{\alpha,k})^{\frac{1}{k+1}}} \tag{2}$$

  From (2), we have

$$\frac{\|D^j u(t)\|^{\frac{1}{j+1}}}{\|D^k u(t)\|^{\frac{1}{k+1}}} \lesssim (T^* - t)^{-\frac{\mu-1}{2\beta}\cdot\frac{k-j}{(j+1)(k+1)}} \frac{(j!)^{\frac{1}{j+1}}}{(k!)^{\frac{1}{k+1}}},$$

The Harmonic-Measure Contraction

Let

$$f(z) = \left(D^{(k)}u\right)_j^{\pm}(x_0 + z\xi, s), \qquad z \in \mathbb{D}_\rho,$$

where ξ is a sparse direction and $\rho \le r_k(s) \le \rho_k(s)$.

- Analyticity yields
$$\sup_{|z|<\rho} |f(z)| \le M.$$
- Sparseness implies that on a subset K of the real diameter with large relative length,
$$|f(x)| \le \lambda M, \qquad x \in K.$$
- Therefore, the harmonic measure maximum principle gives
$$|f(0)| \le \lambda^h M, \qquad h = \omega(0, \mathbb{D}_\rho, K).$$

If x_0 is an extremizing point, then

$$\left\|D^k u(s)\right\| \le \eta \left\|D^k u(t)\right\|, \qquad \eta < 1.$$

The modified bridge for classical Navier–Stokes

Classical viscosity ($\beta = 1$).
Assume that for each $2\ell \le q < k$,

$$\left\|D^q u(t)\right\|_{L^\infty}^{\frac{1}{q+1}} \le \tau_k^{\gamma\left(\frac{1}{(q+1)^\alpha}-\frac{1}{(k+1)^\alpha}\right)} \frac{(q!)^{\frac{1}{q+1}}}{(k!)^{\frac{1}{k+1}}} \left\|D^k u(t)\right\|_{L^\infty}^{\frac{1}{k+1}}.$$

Here $0 < \tau_k < 1$, $0 < \alpha \le 1$, and $\gamma > 0$ controls the strength of the time weight.

Remark

α distributes the gain across orders; γ sets its strength.

Why the classical threshold is one quarter

Set $\Theta = \alpha\gamma(\ell+1)^{1-\alpha}$. Compare the powers in the two scale estimates:

$$\underbrace{\left\|D^q u(t)\right\|_\infty^{-1/(k+1+2\Theta)}}_{\text{analyticity}} \quad \text{and} \quad \underbrace{\left\|D^q u(t)\right\|_\infty^{-1/(k+3/2)}}_{\text{sparseness}}.$$

$$k+1+2\Theta > k+\tfrac{3}{2} \quad \Longleftrightarrow \quad \boxed{\Theta > \tfrac{1}{4}}.$$

The proof tracks the prefactors and evaluates the estimates at the prescribed times.

Application II. The classical continuation theorem

Theorem (Phiri–Grujić (2026))

Assume the bridge and admissibility hypotheses of Theorem 3.1. Suppose

1. for some $\varepsilon > 0$ and every $k \geq K_0 \geq 2\ell$, the function

$$t \longmapsto \left\| D^k(t) \right\|_\infty$$

   is strictly increasing on $(T^* - \varepsilon, T^*)$;
2. $0 < \alpha < 1$, $\gamma > 0$, and $\alpha\gamma(\ell + 1)^{1-\alpha} > 1/4$.

Conclusion

The strong solution extends spatially analytically beyond T^*.

Proof sketch: ruling out blow-up

Assume T^* is the first blow-up time. Choose an escape time t near T^* and a sufficiently large derivative order k.

1. **Escape requires growth.** For every $s \in (t, T^*)$,
$$\|D^k u(s)\|_\infty > \|D^k u(t)\|_\infty.$$
2. **Compare the scales.** Since $\Theta = \alpha\gamma(\ell+1)^{1-\alpha} > \frac{1}{4}$, the bridge gives a positive exponent gap. This gain overcomes the logarithmic factor, giving $r_k(s) \le \rho_k(s)$ at a suitable later time s.
3. **Use sparseness.** The sparse segment lies within the analyticity radius. The harmonic-measure estimate then gives
$$\|D^k u(s)\|_\infty \le \eta \, \|D^k u(t)\|_\infty, \qquad 0 < \eta < 1.$$

This contradicts escape-time growth, so the solution extends past T^*.

Selected References

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